

Wave Packets Propagation in Quantum Gravity

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Abstract

Wave packet broadening in usual quantum mechanics is a consequence of dispersion behavior of the medium which the wave propagates in it. In this paper, we consider the problem of wave packet broadening in the framework of Generalized Uncertainty Principle(GUP) of quantum gravity. New dispersion relations are derived in the context of GUP and it has been shown that there exists a gravitational induced dispersion which leads to more broadening of the wave packets. As a result of these dispersion relations, a generalized Klein-Gordon equation is obtained and its interpretation is given.

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1 Introduction

In probabilistic interpretation of usual quantum mechanics, wave packet broadening is described as reduction of the probability of finding a particle in a given volume at a given time. This reduction of probability or broadening is related to dispersive nature of the medium which wave propagates in it[1]. Consider a more realistic situation which incorporates gravity with quantum theory. This situation has very important candidates such as early Universe and massive black holes interior. In this circumstances the usual uncertainty principle of Heisenberg should be generalized to incorporate gravitational uncertainty from very beginning . This generalized uncertainty principle (or more reasonable terminology of Gravitational Uncertainty Principle) leads to many interesting results in Planck scale physics[2-15]. In this paper we consider the problem of wave packet propagation in GUP. We will show that there is an additional broadening due to gravitational effects and this is a consequence of generalized dispersion relation in GUP. Such a modified dispersion relation leads to a modified form of Klein-Gordon equation. Generalized form of Klein-Gordon equation suggests that one can define a generalized momentum operator and this generalization of momentum operator can be interpreted as generalized De Broglie principle which is the foundation of quantum theory. The other possible interpretation of this generalized momentum operator is related to the fact that \hbar may be a varying "constant" with wave vector. Our calculations show that in Planck scale, the group velocity of the wave packet can be greater than light velocity and this is not surprising since recently variation of fundamental constants of the nature is accepted, at least for fine structure constant which contains both light velocity and \hbar [16-20].

The structure of the paper is as follows: in section 2 first we give a short outline to wave packet propagation in usual quantum mechanics and then our calculation for wave propagation in GUP is given. Section 3 gives a generalized Klein-Gordon equation and a generalized Momentum operator. Summary and Conclusions are given in section 4.

2 Wave Packet Propagation

2.1 Wave Packet Propagation in Ordinary Quantum Mechanics

Consider the following plane wave profile,

$$f(x, t) \propto e^{ikx - i\omega t}. \quad (1)$$

Since $\omega = 2\pi\nu$, $k = \frac{2\pi}{\lambda}$ and $\nu = \frac{c}{\lambda}$, this equation can be written as $f(x, t) \propto e^{ik(x-ct)}$. Now the superposition of these plane waves with amplitude $g(k)$ can be written as,

$$f(x, t) = \int_{-\infty}^{\infty} dk g(k) e^{ik(x-ct)} = f(x - ct) \quad (2)$$

where $g(k)$ can have Gaussian profile. This wave packet is localized at $x - ct = 0$. In the absence dispersion properties for the medium, wave packet will not suffers any broadening with time. In this case the relation $\omega = kc$ holds. In general the medium has dispersion properties and therefore ω becomes a function of wave number, $\omega = \omega(k)$. In this situation equation (2) becomes,

$$f(x, t) = \int dk g(k) e^{ikx - i\omega(k)t}. \quad (3)$$

Suppose that $g(k) = e^{-\alpha(k-k_0)^2}$. With expansion of $\omega(k)$ around $k = k_0$, one find

$$\omega(k) \approx \omega(k_0) + (k - k_0) \left(\frac{d\omega}{dk} \right)_{k_0} + \frac{1}{2} (k - k_0)^2 \left(\frac{d^2\omega}{dk^2} \right)_{k_0}, \quad (4)$$

where using the definitions,

$$\left(\frac{d\omega}{dk} \right)_{k_0} = v_g, \quad \frac{1}{2} \left(\frac{d^2\omega}{dk^2} \right)_{k_0} = \beta, \quad k - k_0 = k'. \quad (5)$$

equation (3) can be written as,

$$\begin{aligned} f(x, t) &= e^{ik_0x - i\omega(k_0)t} \int_{-\infty}^{\infty} dk' e^{-\alpha k'^2} e^{ik'(x-v_g t)} e^{-ik'^2 \beta t} \\ &= e^{ik_0x - i\omega(k_0)t} \int_{-\infty}^{\infty} dk' e^{ik'(x-v_g t)} e^{-(\alpha + i\beta t)k'^2}. \end{aligned} \quad (6)$$

Now completing the square root in exponent and integration gives,

$$f(x, t) = e^{i[k_0x - \omega(k_0)t]} \left(\frac{\pi}{\alpha + i\beta t} \right)^{\frac{1}{2}} e^{-\left[\frac{(x-v_g t)^2}{4(\alpha + i\beta t)} \right]}. \quad (7)$$

Therefore one find,

$$|f(x, t)|^2 = \left(\frac{\pi^2}{\alpha^2 + \beta^2 t^2} \right)^{\frac{1}{2}} e^{-\left[\frac{\alpha(x-v_g t)^2}{2(\alpha^2 + \beta^2 t^2)} \right]}, \quad (8)$$

which is the profile of the wave in position space. The quantity which in $t = 0$ was α , now has became $\alpha + \frac{\beta^2 t^2}{\alpha}$ and this is the notion of broadening. Therefore,

$$Broadening \propto \left(1 + \frac{\beta^2 t^2}{\alpha^2} \right)^{\frac{1}{2}}. \quad (9)$$

This relation shows that a wave packet with width $(\Delta x)_0$ in $t = 0$ after propagation will have the following width,

$$(\Delta x)_t = (\Delta x)_0 \left(1 + \frac{\beta^2 t^2}{\alpha^2} \right)^{\frac{1}{2}}. \quad (10)$$

2.2 Wave Packet Propagation in Quantum Gravity

As has been indicated, when one considers gravitational effects, usual uncertainty relation of Heisenberg should be replaced by,

$$\Delta x \geq \frac{\hbar}{\Delta p} + \frac{\alpha' l_p^2 \Delta p}{\hbar}. \quad (11)$$

It is important to note that there are more generalization which contain further terms in right hand side of equation (11) (see [14]), but in some sense equation (11) has more powerful physical grounds. So as a first step analysis we consider the above simple form of GUP. Suppose that

$$\Delta x \sim x, \quad \Delta p \sim p, \quad p = \hbar k, \quad x = \bar{\lambda} = \frac{\lambda}{2\pi}.$$

Therefore one can write,

$$\bar{\lambda} = \frac{1}{k} + \alpha' l_p^2 k \quad \text{and} \quad \omega = \frac{c}{\bar{\lambda}}. \quad (12)$$

In this situation the dispersion relation becomes,

$$\omega = \omega(k) = \frac{kc}{1 + \alpha' l_p^2 k^2}. \quad (13)$$

This relation can be described in another viewpoint. By expansion of $\left(1 + \alpha' l_p^2 k^2\right)^{-1}$ and neglecting second and higher order terms of α' , we find that $\omega = kc(1 - \alpha' l_p^2 k^2)$. This can be considered as $\omega = k'c$ where $k' = k(1 - \alpha' l_p^2 k^2)$. Now one can define a generalized momentum as $p = \hbar k' = \hbar k(1 - \alpha' l_p^2 k^2)$. It is possible to consider this equation as $p = \hbar' k$ where $\hbar' = \hbar(1 - \alpha' l_p^2 k^2)$. So one can interpret it as a wave number dependent Planck "constant". In the same manner group velocity becomes,

$$v_g = \left. \frac{d\omega}{dk} \right|_{k=k_0} = \left. \frac{c(1 - \alpha' l_p^2 k^2)}{(1 + \alpha' l_p^2 k^2)^2} \right|_{k=k_0}. \quad (14)$$

Up to first order in α' this relation reduces to $v_g \approx c(1 - 3\alpha' l_p^2 k_0^2)$.

A little algebra gives β as follow

$$\beta = \left. \frac{1}{2} \left(\frac{d^2 \omega}{dk^2} \right) \right|_{k=k_0} = \left. \frac{-3\alpha' l_p^2 c k (1 + \alpha' l_p^2 k^2)^2 + 4\alpha'^2 l_p^4 c k^3 (1 + \alpha' l_p^2 k^2)}{(1 + \alpha' l_p^2 k^2)^4} \right|_{k=k_0}, \quad (15)$$

which up to first order in α' reduces to $\beta \approx -3\alpha' l_p^2 c k_0$. It is evident that when $\alpha' \rightarrow 0$ then $\beta \rightarrow 0$ and $v_g \rightarrow c$. The same analysis which has leads us to equation (10), now gives the following result,

$$(\Delta x)_t = (\Delta x)_0 \left(1 + \frac{1}{\alpha^2} \left(\frac{-3\alpha' l_p^2 c k_0 (1 + \alpha' l_p^2 k_0^2)^2 + 4\alpha'^2 l_p^4 c k_0^3 (1 + \alpha' l_p^2 k_0^2)}{(1 + \alpha' l_p^2 k_0^2)^4} \right)^2 t^2 \right)^{\frac{1}{2}}. \quad (16)$$

If one accepts that α' is negative constant ($\alpha' < 0$), then group velocity of the wave packet becomes greater than light velocity. This is evident from equation (14) and is reasonable from varying speed of light models. In fact if $|\alpha'| k^2 l_p^2 \ll 1$, one recover usual quantum mechanics but when $|\alpha'| k^2 l_p^2 \approx 1$, Planck scale quantum mechanics will be achieved. Based on this argument, equation (16) shows that in quantum gravity there exists a more broadening of wave packet due to gravitational effects. Up to first order in α' , this equation becomes,

$$(\Delta x)_t = (\Delta x)_0 \left(1 - \frac{3\alpha' l_p^2 c k_0 t^2}{\alpha^2} \right)^{\frac{1}{2}}. \quad (17)$$

Now using equation (13), one can write the dispersion relation as the following form also,

$$\omega(p) = \frac{\hbar p c}{\hbar^2 + \alpha' l_p^2 p^2}, \quad (18)$$

or

$$E' = \hbar \omega(p) = \frac{p c}{1 + \alpha' \left(\frac{l_p p}{\hbar} \right)^2}. \quad (19)$$

It is evident that if $\alpha' \rightarrow 0$ Then $E' \rightarrow E = p c$ and $\omega(p) \rightarrow \omega = \frac{p c}{\hbar}$. These dispersion relations provide a framework for definition of generalized momentum operator and generalized Klein-Gordon equation.

3 A Generalized Klein-Gordon Equation

Now consider the following integral which has been used heuristically by Schrödinger to find his equation[1],

$$\psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp \phi(p) e^{i(px - E't)/\hbar}. \quad (20)$$

Differentiation twice relative to x gives,

$$\frac{\partial^2 \psi(x, t)}{\partial x^2} = -\frac{1}{\hbar^2} \times \frac{1}{\sqrt{2\pi\hbar}} \int dp p^2 \phi(p) e^{i(px - E't)/\hbar}, \quad (21)$$

while differentiation relative to t leads to

$$\frac{\partial^2 \psi(x, t)}{\partial t^2} = -\frac{c^2}{\hbar^2} \times \frac{1}{\sqrt{2\pi\hbar}} \int dp \left(\frac{p}{1 + \alpha' \left(\frac{l_p p}{\hbar} \right)^2} \right)^2 \phi(p) e^{i(px - E't)/\hbar}. \quad (22)$$

It is evident that if $\alpha' \rightarrow 0$, then $\frac{\partial^2 \psi(x, t)}{\partial t^2} = c^2 \frac{\partial^2 \psi(x, t)}{\partial x^2}$ which is the usual wave equation. Now expansion of integrand in equations (19) and (20) gives,

$$\begin{aligned} \frac{\partial^2 \psi(x, t)}{\partial t^2} &= -\frac{c^2}{\hbar^2} \times \frac{1}{\sqrt{2\pi\hbar}} \int dp p^2 \left(1 - \alpha' \left(\frac{l_p p}{\hbar} \right)^2 + O(\alpha'^2) - \dots \right)^2 \phi(p) e^{i(px - E't)/\hbar} \\ &= -\frac{c^2}{\hbar^2} \times \frac{1}{\sqrt{2\pi\hbar}} \int dp \left(p^2 - \frac{2\alpha' l_p^2}{\hbar^2} p^4 + O(\alpha'^2) + \dots \right) \phi(p) e^{i(px - E't)/\hbar}. \end{aligned} \quad (23)$$

The first order terms in α' , satisfy the following equation,

$$\frac{\partial^2 \psi(x, t)}{\partial t^2} = c^2 \frac{\partial^2 \psi(x, t)}{\partial x^2} - 2\alpha' l_p^2 c^2 \frac{\partial^4 \psi(x, t)}{\partial x^4}. \quad (24)$$

This is a generalized wave equation and in some sense can be considered as generalized Klein-Gordon equation for a massless particle. The solution of this equation gives the correct profile of the wave in Planck scale. Adler and Santiago in their paper[21] have indicated that: "It has long been a supposed fact of life that the differential equations of physics are first or second order. This is well born out by experiences in that classical mechanics, classical electromagnetism, general relativity, non-relativistic quantum mechanics, relativistic quantum mechanics, and all the equations of the standard model of particles are at most of second order. But it may well be that we also have developed an unjustified bias in favor of second order equations due to mathematical convenience. As such it is particularly interesting to consider higher order equations with all their inherent dangers and difficulties with boundary conditions". In our opinion equation (22) is one of the mentioned higher derivative equations. For a massive particle(or field), equation (22) can be written as,

$$-\hbar^2 \frac{\partial^2 \psi(x, t)}{\partial t^2} = \left(-\hbar^2 c^2 \frac{\partial^2}{\partial x^2} + 2\alpha' \hbar^2 l_p^2 c^2 \frac{\partial^4}{\partial x^4} + m_0^2 c^4 \right) \psi(x, t). \quad (25)$$

Note that this is only the first order approximation equation since we have considered only the first order term in (22).

In the language of operators, usual Klein-Gordon equation can be written as,

$$-\hbar^2 \frac{\partial^2 \psi(x, t)}{\partial t^2} = \left(c^2 p_{op}^2 + m_0^2 c^4 \right) \psi(x, t), \quad (26)$$

where $p_{op} = \frac{\hbar}{i} \frac{\partial}{\partial x}$. Now the generalized momentum operator in GUP up to first order in β' , takes the following form,

$$p_{op}^{(GUP)} = \frac{\hbar}{i} \left(1 + \beta' \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 \right) \frac{\partial}{\partial x}, \quad (27)$$

or

$$\left(p_{op}^{(GUP)} \right)^2 = -\hbar^2 \frac{\partial^2}{\partial x^2} + 2\beta' \hbar^4 \frac{\partial^4}{\partial x^4} \quad (28)$$

and this operator immediately gives (25) with $\beta' = \alpha' \frac{l_p^2}{\hbar^2}$. The generalization to 3-dimensional case is straightforward. It is important to note that one can consider from the beginning a generalized form of momentum operator. In this case equation $p = \hbar k$ should be replaced by $p = \hbar k'$ where $k' = k(1 - \alpha' l_p^2 k^2)$. In this situation equation (20) should be modified by this generalized momentum and we find ordinary Klein-Gordon equation but now with generalized operators.

4 Summary and Conclusions

In this paper we have shown that:

- 1- In quantum gravity, wave packet broadening is more than corresponding broadening of wave packet in usual quantum mechanics because of inherent gravitational uncertainties.
- 2- There are generalized dispersion relations in quantum gravity which lead to generalized group velocity, generalized momentum operator and generalized Klein-Gordon equation.
- 3- One can describe the generalized dispersion relations being as a consequence of generalized de Broglie principle or varying Planck constant(as a function of momentum).
- 4- Our analysis shows that gravitational uncertainty principle leads to varying speed of light scenario in such a way that speed of light in early universe was greater than its present value.
- 5- It seems that in Planck scale quantum mechanics, higher order derivatives will be appear in equations. This is seen in generalized Klein-Gordon equation and is a result of generalized momentum operator.

- 6- Our approach provides foundations for constructing a gravitational quantum mechanics. This is the subject of our forthcoming work.

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